Extension of Gyro-Landau Fluid Equations to Higher Order For Application to Edge Plasmas*

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Abstract: 1

The edge of a magnetic confinement fusion reactor connects the hot collisionless plasma in the core to the cold collisional plasma near the divertor targets. The near-separatrix region is challenging to model because of large edge gradients and flows, large magnetic shear and complex geometry, and large nonlinear fluctuations. In this work, gyro-Landau fluid (GLF) theory, which is based on the assumption that the distribution function is close to Maxwellian, is extended to more accurately treat the effects of finite collisionality and finite perturbation amplitude for applications to edge plasmas.

Previously, gyro-kinetic (GK) theory was extended [1] to treat the case of large nonlinearities in potential, subject to the restriction $(k\rho)^2 e\delta\psi/T_0 \sim \mathcal{O}(1)$. Large gradients and flows can be treated by retaining the perpendicular kinetic energy of gyro-averaged drifts in the 1st order Hamiltonian. As usual, the Hamiltonian is modified at 2nd order by quadratic correlations in the gyro-phase dependent part of potential, e.g. $-\partial\langle\delta\psi^2\rangle/2\Omega\partial\mu$. Taking fluid moments of these terms mixes spatial degrees of freedom through integration of FLR effects over velocity space, leading to "nonlinear phase mixing." In the isothermal case, the quadratic terms can be expressed exactly through a bilinear 4D spatial integral operator. Ref. [3] found that two successive linear 2D spatial integral operators $\Gamma_0^{1/2}((k\rho)^2)$ yield a reasonable approximation for $k\rho \sim \mathcal{O}(1)$.

[1] A. M. Dimits, Phys. Plasmas 19, 022504 (2012).
[2] P. SImilon, Ph.D. Thesis, Princeton University (1981).
[3] W. Dorland and G. W. Hammett, Phys. Fluids B 5, 812 (1993).

Abstract: 2

Here, it is shown that the accuracy can be improved by including additional 2nd order perpendicular derivative operators in the Padé approximation, leading to a nonlinear phase-mixing closure that differs from [3]. In order for the system to conserve energy, the Poisson equation must also retain quadratic terms, e.g. in density and potential. These quadratic terms can be expressed through the adjoint bilinear 4D spatial integral operator and a similar successive approximation using 2D operators can be derived.

Collisionless Landau closures based on fitting linear dispersion relations for the core plasma [4] typically retain a large number of moments (typically 4-6) for accuracy in both linear and nonlinear physics, but neglect finite collisionality and nonlinearity in the closure itself. A more general closure technique was developed in [5] to treat the (3 moment) Chapman-Enskog fluid expansion without fitting a linear dispersion relation. In principle, this technique can be generalized to accurately treat nonlinearity and collisions for arbitrary numbers of moments. Because this approach is generalizable, it can be used to yield a definite prediction for toroidal closure terms, generalizing the "toroidal closures" in [6]. In the future, this technique may be used to accurately treat other effects such as collisionless banana orbits and nonlinear collisional interactions.

[4] G. W. Hammett and F. W. Perkins, Phys. Rev. Lett. 64, 3019 (1990).
[5] Z. Y. Chang and J. D. Callen, Phys. Fluids B 4, 1167 (1992).
[6] M. A. Beer and G. W. Hammett, Phys. Plasmas 3, 4046 (1996).

Outline & Summary

- **Gyro-Landau fluid (GLF) equations can be extended to higher order nonlinear effects in a similar manner as gyro-kinetics (GK)**
  - FLR effects in extended GK ordering are similar to usual form
  - 3rd order Hamiltonian yields 2nd order terms in equations of motion and in polarization

- **Perpendicular Hierarchy**
  - As a first step, the electrostatic isothermal case is investigated here
  - Extension of Hamiltonian to higher order must be consistent with nonlinear polarization effects
  - A new form for nonlinear polarization effects is derived that is accurate for $k\rho \sim 1$

- **Parallel Hierarchy**
  - Requires many $\sim \mathcal{O}(\omega_t/\nu)$ moments for accurate linear distribution function near a resonance
  - A proof that the correct linear kinetic response (Hammet-Perkins approach) can be achieved with a specific linear projection of kinetic equation is given (extension of Chang-Callen approach)
  - In the future, this technique will extended to include new & nonlinear physics effects in a straightforward manner

- **Linear Polarization in Extended Gyrokinetics**
  - Moving Frame Formalism (Dimits 2010 POP)

Nonlinear Extension of Perpendicular Gyro-Fluid Equations

Linear Gyrokinetic Poisson Equation

- **The 1st order transformation induces the response**

$$n_e = n_i = \Gamma_0^{1/2} n_{ig} + n_0 \left((\Gamma_0 - 1) e\phi/T_i \right)$$

  - **Solving for the quasineutral potential**

$$(\Gamma_0 - 1) e\phi/T_i = (n_e - \Gamma_0^{1/2} n_{ig})/n_0$$

  - **Including parallel Boltzmann electron response, the Poisson equation becomes**

$$(\Gamma_0 - 1) e\phi/T_i + e \left\{\phi\right\}_{\|} /T_e = (n_{eb} - \Gamma_0^{1/2} n_{ig})/n_0$$

  - **Including parallel Boltzmann ion response as well yields**

$$n_e = n_i = \Gamma_0^{1/2} n_{ib} + n_0 \left((\Gamma_0 - 1) e\phi/T_i - e\{\phi\}_{\|}/T_i \right)$$

  - **and the Poisson equation**

$$(\Gamma_0 - 1) e\phi/T_i - e\{\phi\}_{\|}/T_i + e \left\{\phi\right\}_{\|} /T_e = (n_{eb} - \Gamma_0^{1/2} n_{ib})/n_0$$

- **Definition:** Field line averaging operations

$$\langle X \rangle_{\|} = \frac{\int X d\ell/B}{\int d\ell/B}.$$

$$\{X\}_{\|} = X - \langle X \rangle$$

Hamiltonian Approach to GyroFluid Equations

- **Gyrokinetic GK Hamiltonian to 2nd order**

$$\mathcal{H} = \int d^6 Z \left(\tfrac{1}{2} m u^2 + \mu B + e_i \bar{\phi} + \frac{e_i^2}{2 m_i \Omega_i} \left(\partial_\mu \tilde{\phi}^2 + \hat{\mathbf{b}} \cdot \nabla \tilde{\phi} \times \nabla \tilde{\Phi} \right) \right) F$$

$$\bar{\phi} = J_0 \phi$$

$$\tilde{\phi}(\gamma) = 1 - \bar{\phi}$$

$$\tilde{\Phi}(\gamma) = \int_0^\gamma \tilde{\phi}(\gamma') d\gamma' / \Omega_i$$

- **Can be promoted to Gyro-fluid (GF) Hamiltonian to 2nd order**

$$\mathcal{H} = \int d^3 R \left(\tfrac{1}{2}(m N U^2 + 3P) + N \langle J_0 \rangle e_i \phi + \frac{e_i^2}{2T_0} \phi \left[\bar{\Gamma}_0(N, \phi) - \bar{\Upsilon}_0(N, \phi) \right] \right)$$

$$\simeq \int d^3 R \left(\tfrac{1}{2} m U^2 + \tfrac{3}{2} T + \Gamma_0^{1/2} e_i \phi + \frac{e_i^2}{2T_0} \left[(\Gamma_0^{1/2} \phi)^2 - \phi^2 \right] \right) N$$

- **Definition & lowest order Padé approximation (scaled to ρ)**

$$\bar{\Gamma}_0(A, B) = \int \Gamma_0(\mathbf{k}_\perp, \mathbf{q}_\perp) A_{k-q} B_q e^{i \mathbf{k}_\perp \cdot \mathbf{x}} d^2 k_\perp d^2 q_\perp$$

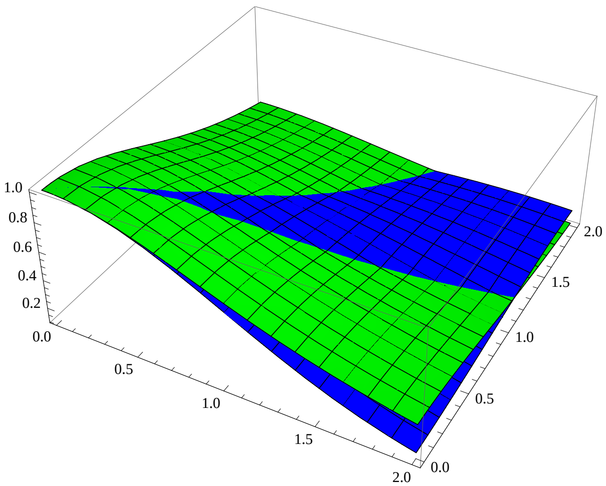
$$\Gamma_0(\mathbf{k}, \mathbf{q}) = I_0(|k||q|) e^{-(k^2 + q^2)/2}$$

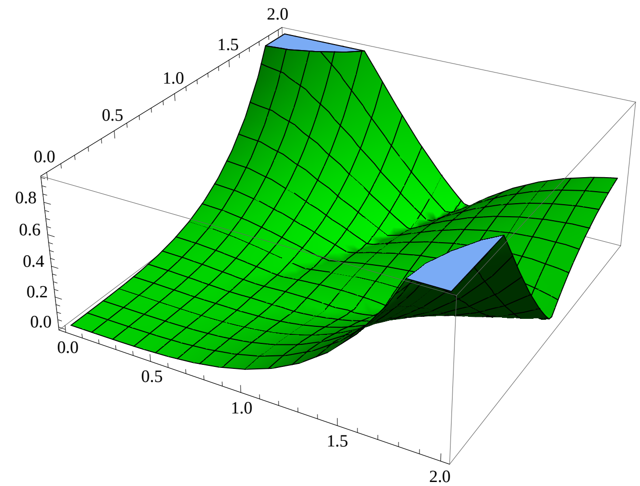
$$\bar{\Upsilon}_0(A, B) = \int \Upsilon_0(\mathbf{k}_\perp, \mathbf{q}_\perp) A_{k-q} B_q e^{i \mathbf{k}_\perp \cdot \mathbf{x}} d^2 k_\perp d^2 q_\perp$$

$$\Upsilon_0(\mathbf{k}, \mathbf{q}) = e^{-(|k| - |q|)^2/2}$$

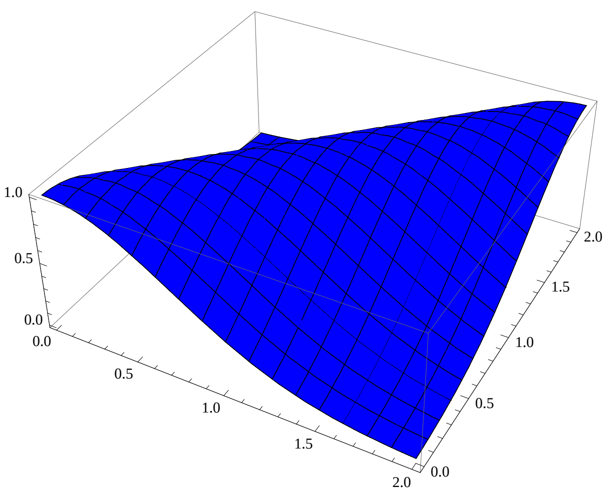
Padé Approximations for $\Gamma_0(k,q)$ and $\Upsilon_0(k,q)$

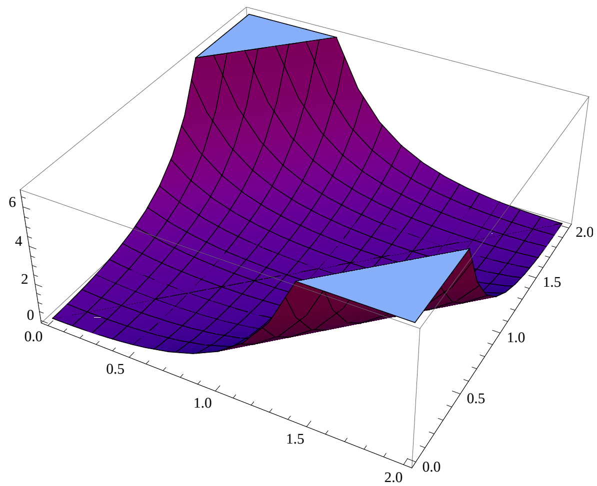
$$\Gamma_0(k,q) \simeq \Gamma_0^{1/2}(k^2)\Gamma_0^{1/2}(q^2) = 1/(1+k^2/2)(1+q^2/2) \quad (1)$$





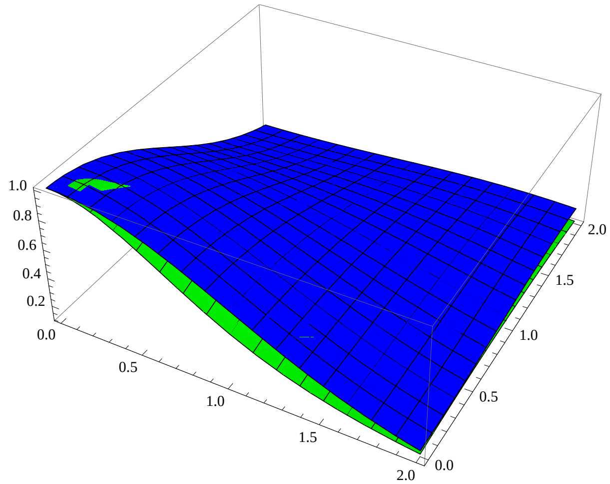
$$\Upsilon_0(k,q) \simeq 1 \quad (2)$$

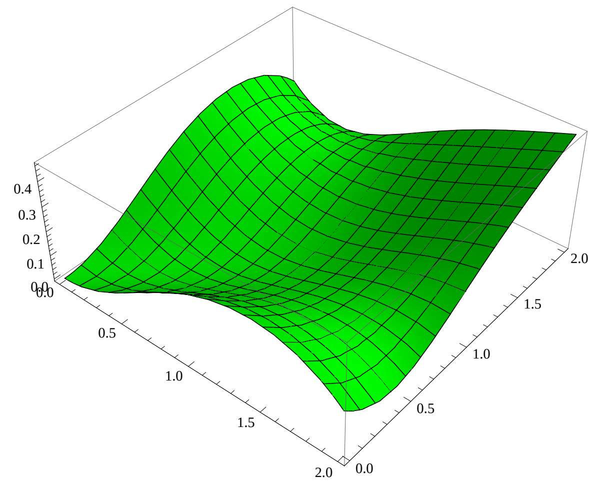




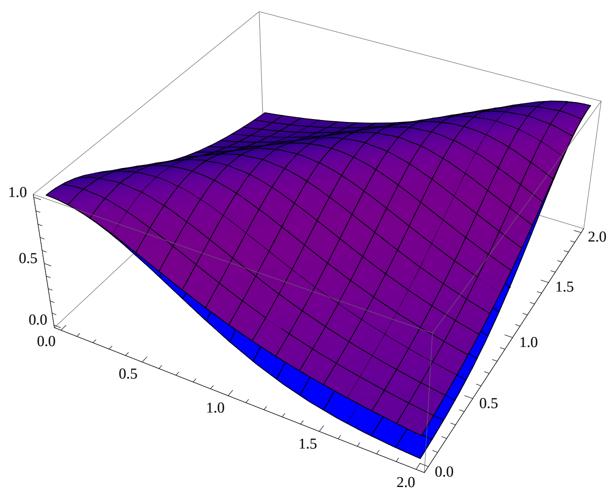
Improved approximation for $\Gamma_0(k,q)$ and $\Upsilon(k,q)$

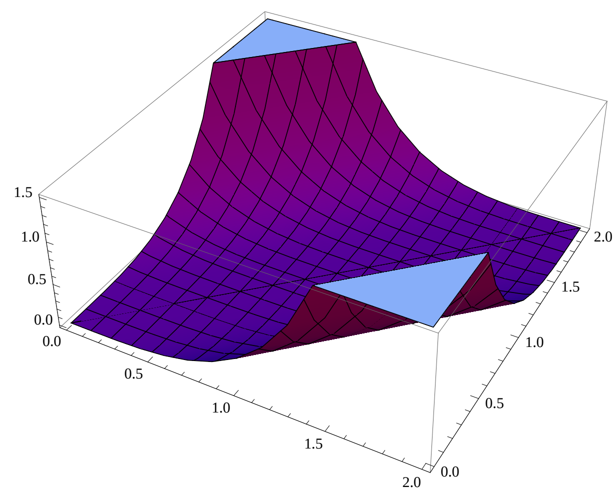
$$\Gamma_0(k,q) \simeq \Upsilon_0(k,q)\Gamma_0^{1/2}(k^2)\Gamma_0^{1/2}(q^2) = 1/(1+(|k|-|q|)^2)(1+k^2/2)(1+q^2/2) \tag{3}$$



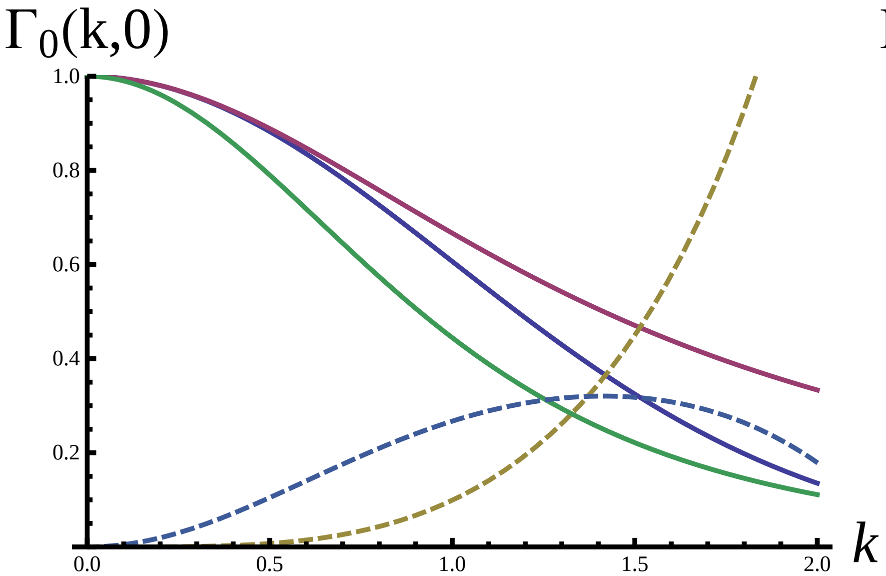


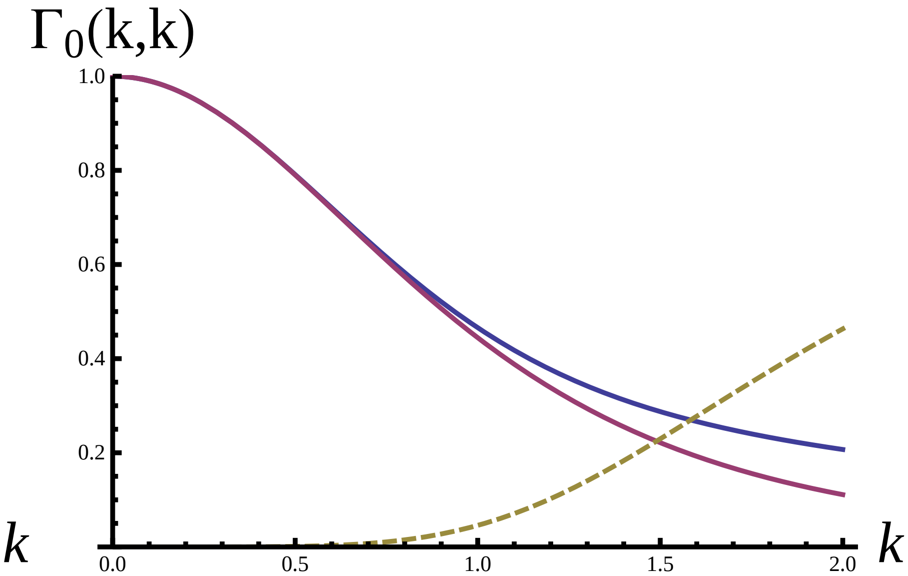
$$\Upsilon_0(k,q) \simeq 1/(1+(|k|-|q|)^2) \tag{4}$$

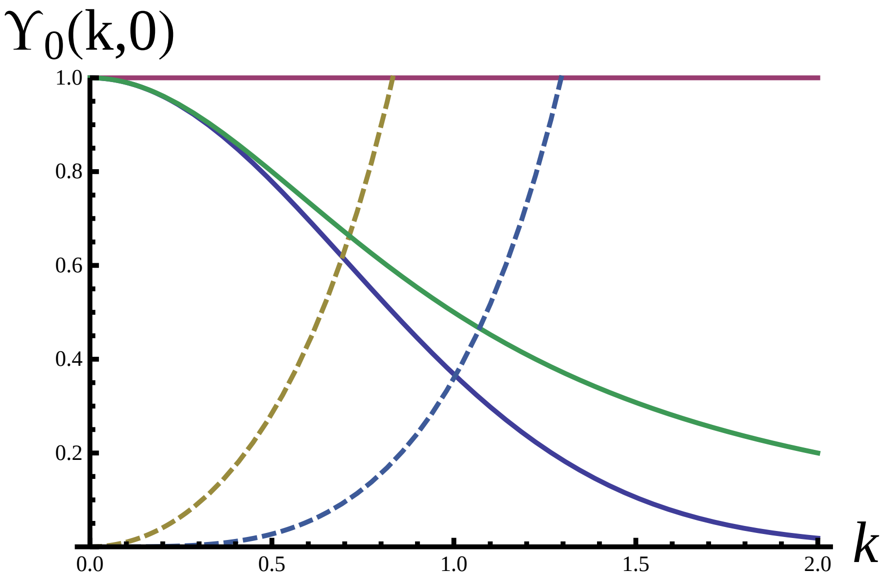




Detailed comparison of the approximations for $\Gamma_0(k,q)$ and $\Upsilon_0(k,q)$







Nonlinear Quasineutral Poisson Equation $n_e = n_i$ Using Simple Approximations

- **Matches known result for $n_{ig} = n_0 + \delta n_{ig}$**

$$T_{i\perp}^{-1}\left(\Gamma_0^{1/2} \circ \left(n_{ig}\Gamma_0^{1/2} e\phi\right) - en_{ig}\phi\right) = n_e - \Gamma_0^{1/2} n_{ig} \tag{5}$$

$$T_{i\perp}^{-1}\left(n_{ig}\Gamma_0 e\phi + \nabla n_{ig}\cdot\nabla(\Gamma_0 - \Gamma_1)e\phi + \ldots\right) \simeq n_e - \Gamma_0^{1/2} n_{ig} \tag{6}$$

- **Approximate quasi-neutral Poisson equation including Boltzmann electron response**

$$T_{i\perp}^{-1}\left(\Gamma_0^{1/2} \circ \left(n_{ig}\Gamma_0^{1/2} e\phi\right) - n_{ig}e\phi\right) + T_{e,\|}^{-1}\{e\phi\}_{\|} = n_{eb} - \Gamma_0^{1/2} n_{ig} \tag{7}$$

- **Even more accurate**

$$T_{i\perp}^{-1}\Upsilon_0\left(\Gamma_0^{1/2} \circ \left(n_{ig}\Gamma_0^{1/2} e\phi\right) - en_{ig}\phi\right) = n_e - \Gamma_0^{1/2} n_{ig} \tag{8}$$

Equations of Motion

- **Ion GK equation**

$$\partial_t F_i + \nabla \cdot \left(u\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla \bar{\psi}_i}{B} \right) F_i - \partial_u \hat{\mathbf{b}} \cdot \nabla \frac{e_i \psi_i}{m_i} F_i = 0$$

- **Gyro-averaged Potential to 2nd order**

$$\bar{\psi}_i = J_0 \phi + \sum_{\ell \neq 0} \tfrac{e_i}{2m_i \Omega_i} \left(\partial_\mu J_\ell \phi J_\ell \phi + \hat{\mathbf{b}} \cdot \nabla J_\ell \phi \times \nabla \frac{J_\ell \phi}{i\ell \Omega_i} \right)$$

- **Maxwellian-averaged Potential to 2nd order**

$$\bar{\Psi}_i = \Gamma_0^{1/2} \phi + \tfrac{e_i}{T_{\perp i}} \left(\nabla \phi_0 \cdot \nabla \Gamma_0^{1/2} \phi_1 + \left(\Gamma_0^{1/2} \phi_1 \right)^2 - \phi_1^2 \right)$$

- **Ion GF equation**

$$\partial_t N_i + \nabla \cdot N_i \left(U\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla \bar{\Psi}_i}{B} \right) = 0$$

$$\partial_t N_i U + \nabla \cdot \left(N_i \left(U^2 + \frac{T_{\parallel i}}{m_i} \right) \hat{\mathbf{b}} + N_i U \frac{\hat{\mathbf{b}} \times \nabla \bar{\Psi}_i}{B} \right) = -N_i \hat{\mathbf{b}} \cdot \nabla \frac{e_i \bar{\Psi}_i}{m_i}$$

Gyro-vorticity $\varpi = en_e - e_i N_i$ & Gyro-density (n_e, N_i) Formulation

- **Electron density equation**

$$\partial_t e n_e + \nabla \cdot e n_e \left(U\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla\phi}{B} \right) = -\nabla \cdot J\hat{\mathbf{b}}$$

- **Ion Gyro-density equation**

$$\partial_t e_i N_i + \nabla \cdot e_i N_i \left(U\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla\bar{\Psi}_i}{B} \right) = 0$$

- **Gyro-vorticity equation should be used in one of these forms:**

$$\partial_t \varpi + \nabla \cdot \varpi \left(U\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla\phi}{B} \right) + \nabla \cdot e_i N_i \frac{\hat{\mathbf{b}} \times \nabla(\phi - \bar{\Psi}_i)}{B} = -\nabla \cdot J\hat{\mathbf{b}}$$

$$\partial_t \varpi + \nabla \cdot \varpi \left(U\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla\bar{\Psi}_i}{B} \right) + \nabla \cdot e n_e \frac{\hat{\mathbf{b}} \times \nabla(\bar{\Psi}_i - \phi)}{B} = -\nabla \cdot J\hat{\mathbf{b}}$$

Gyro-vorticity & Gyro-pressure Formulation

- **Defining relations**

$$
\begin{aligned}
P &= n_e T_e + N_i T_i \\
\varpi &= e n_e - e_i N_i \\
n_e &= (e_i P + \varpi T_i)/(e_i T_e + e T_i) \\
n_i &= (e P - \varpi T_e)/(e_i T_e + e T_i) \\
T_s &= T_e + T_i e/e_i \\
\tau_i &= e T_i/(e_i T_e + e T_i) = e T_i/e_i T_s
\end{aligned}
$$

- **Gyro-pressure** $P = T_e n_e + T_i N_i$ **equation**

$$
\partial_t P + \nabla \cdot P \left(U\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla(\phi(1-\tau_i) + \bar{\Psi}_i \tau_i)}{B} \right) + \nabla \cdot \tau_i \frac{\varpi T_e}{e} \frac{\hat{\mathbf{b}} \times \nabla(\phi - \bar{\Psi}_i)}{B} = -\nabla \cdot J\hat{\mathbf{b}}
$$

- **Gyro-vorticity** $\varpi = e n_e - e_i N_i$ **equation**

$$
\partial_t \varpi + \nabla \cdot \varpi \left(U\hat{\mathbf{b}} + \frac{\hat{\mathbf{b}} \times \nabla(\bar{\Psi}_i(1+\tau_i) - \phi)}{B} \right) + \nabla \cdot \frac{eP}{T_s} \frac{\hat{\mathbf{b}} \times \nabla(\bar{\Psi}_i - \phi)}{B} = -\nabla \cdot J\hat{\mathbf{b}}
$$

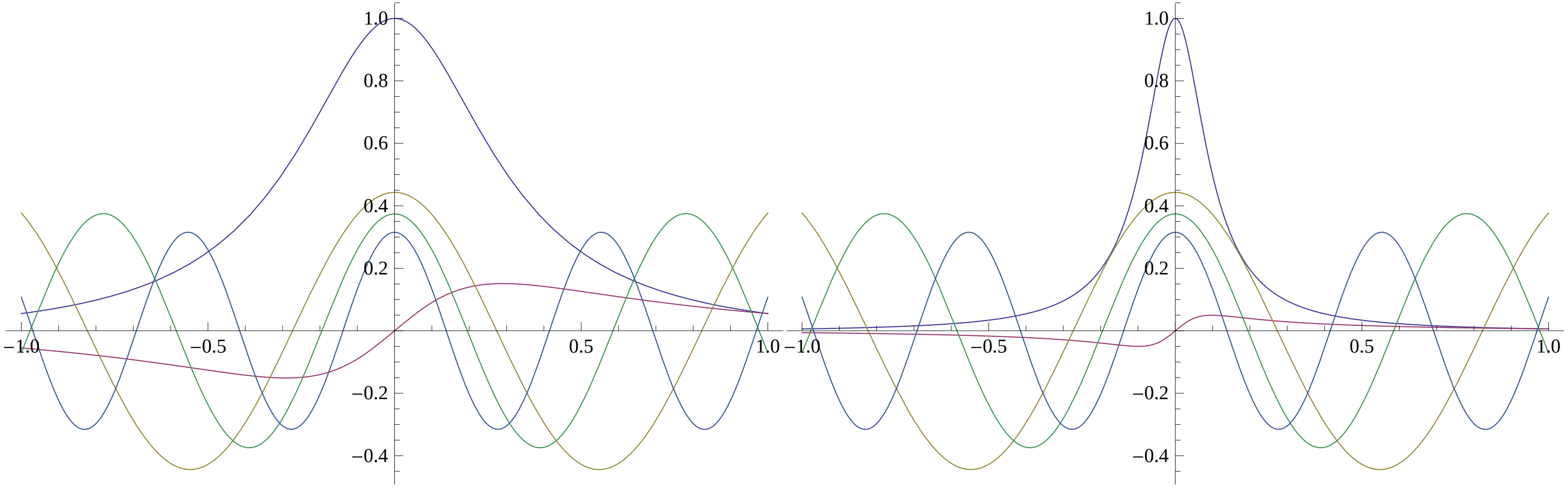
Nonlinear Extension of Parallel Landau-fluid Equations

The Parallel Closure Problem

- **Is there an accurate closure for the moment hierarchy?** $V_T = \sqrt{T/m}$, $m = 1$

$$
\begin{aligned}
\partial_t n + \nabla \cdot nv &= 0 \\
\partial_t nv + \nabla \cdot nvv + \nabla p &= qnE \\
\partial_t p + \nabla \cdot (vp + q) + (\gamma - 1) p \nabla v &= 0 \\
\partial_t q + \nabla \cdot (vq + r) + 3q\nabla v - 3V_T^2 \nabla p + \nu_3 q &= 0 \\
\ldots
\end{aligned}
$$

- Moment equations have a sparse structure, which is useful for numerical accuracy
- However, moments are NOT a sparse representation for f (see J.Y. Ji and E. Held PoP (2009)



Hammett-Perkins Algorithm

- **Rational fit to linear response function using** $\sigma_\nabla \equiv \frac{\nabla}{|\nabla|}$

$$
\begin{aligned}
q_3 &= -2\sqrt{\frac{2}{\pi}} n V_T \sigma_\nabla T \\
r_4 &= 3pT + \delta r_4 \\
\delta r_4 &= \frac{32-9\pi}{3\pi-8} p\tilde{T} - \frac{2\sqrt{2\pi}}{3\pi-8} V_T \sigma_\nabla q \\
? &= \frac{32-9\pi}{3\pi-8} \sigma_\nabla p \sigma_\nabla T - \frac{2\sqrt{2\pi}}{3\pi-8} V_T \sigma_\nabla q
\end{aligned}
$$

The Closure Problem Returns

- **How can this be extended to nonlinear physics? new physics?**

$$\partial_t q = 0$$

$$\left[\nabla \cdot \left(v - \frac{2\sqrt{2\pi}}{3\pi - 8} V_T \sigma_\nabla\right) + \nu_3\right] q + \frac{8}{3\pi - 8} p \nabla T + 3 q \nabla v \simeq 0$$

- If $\nu, v \to 0$ but $\nabla v \neq 0$

$$q \to -\sigma_\nabla \frac{B}{2\sqrt{2\pi} V_T} \int \frac{d\ell}{B} \left(8 p \nabla T + 3(3\pi - 8) q \nabla v\right)$$

- If $v \to 0$ but $\nabla v \neq 0$, let $\hat{\nu}_3 = \frac{3\pi - 8}{2\sqrt{2\pi}}$

$$q \to M^\dagger (M M^\dagger)^{-1} \frac{1}{2\sqrt{2\pi}} \left(8 p \nabla T + 3(3\pi - 8) q \nabla v\right)$$

$$M = -\nabla \cdot V_T \sigma_\nabla + \hat{\nu}_3$$

$$M M^\dagger = \hat{\nu}_3^2 - \nabla \cdot \frac{8\pi V_T^2}{(3\pi - 8) B} \nabla B + (M + M^\dagger)\hat{\nu}_3 \simeq \hat{\nu}_3^2 + \frac{8\pi V_T^2}{(3\pi - 8)} k_\parallel^2$$

- Terms will match if extended to higher order, but this method is only linearly accurate

A Natural Galerkin Approach

- **The kinetic equation contains accurate nonlinear physics through the nonlinear collision operator**

$$KE[f] = Df - C[f] = S$$
$$Df = (\partial_t + \nabla_x \cdot v + \nabla_v \cdot qE/m) f$$

- **The Galerkin method is a natural technique for discretizing the equations**
  - Choose a linearly independent basis $e_n(v)$ and a Hermitian inner product $\langle w|e\rangle$ to define a dual basis $w_n(v)$

$$\langle w|e\rangle = \int weJ(x, v, t)dv$$
$$\langle w_n|e_m > = \delta_{nm}$$

  - This basis is complete over it's span & admits representation

$$f = \sum_n f_n(x, t)e_n(v)F(v, x, t)$$
$$W \equiv J/F.$$

  - Discretization of the nonlinear equations proceeds via projection

$$FE_n = S_n$$
$$FE_n = \int w_n KE[f]W dv$$
$$S_n = \int w_n SW dv.$$

- **We seek a choice of weights that yield accurate linear and nonlinear physics**

Pade Approximation for Linear Response

- **The linearized equations must have the form**

$$FE_n = \sum_m (M_{nm}\partial_t + L_{nm}\partial_x + N_{nm}) f_m = \sum_j K_{nj} s_j.$$

  - This matrix provides a Padé approximation for eigenvectors & eigenvalues
  - High frequency response determined by M
  - Low frequency response determined by L
- **The linear constraints critically depend on the assumed form of the source functions**
  - For the same basis as f, one can use completeness of the basis and $M = K$ in proofs of following

$$S = \sum_n s_n(x,t) e_n(v)$$

- **Response function for Galerkin truncation**
  - High frequency response is automatically correctly satisfied (using completeness)

$$-i\omega M R_{\omega\to\infty} = K = M$$

  - Low frequency response is only consistent with kinetic result for a specific choice of weights

$$(ikL + N) R_{\omega\to 0} = K = M$$

Padé Approximation Completely Determined by Low-frequency Response

- **Determined from solution to kinetic equation**

$$f = S/(ikv + \nu)$$
$$f_n = \langle w_n \left|(ikv + \nu)^{-1}\right| S/F >$$

- **Thus, if the weight scales as $1/(ikv + \nu)$, the correct kinetic response function will be achieved!**
- Here, we will only consider non-resonant response needed for linear physics; the full nonlinear approach must eventually consider resonant particles
- **Nonresonant response can be treated exactly in 2 limits**
- **High frequency $\omega \gg ikv, \nu$ (Chang-Callen)**
  - Distribution function f is a polynomial expansion of initial $F = F_0$
  - Use Galerkin weight $W = 1/(ikv - \nu)$
- **Low frequency $\omega \ll ikv, \nu$ (New!)**
  - Distribution function f is a polynomial expansion of initial $F = F_0/(ikv - \nu)$
  - This basis consists of polynomials & only 1 singular basis function

$$e_n = \left(\frac{ikv_N}{ikv - \nu} - \sum_{j=0}^{n-1} Ze_j \right) F_0$$

  - Use Galerkin weight $W = 1$

Solution to Parallel Closure Problem

- **First n basis functions e_m for $m = 0, ...n-1$ are usual polynomials**
- **Simply use original moment equations**
- **Final coefficient f_n must be treated with the closure equation for the basis function e_n**

$$\sum_m M_{nm} \partial_t f_m + (\partial_x v_N + \nu_n) f_n = \sum_m M_{nm} s_m.$$

- **Substitution of previous n moment equations yields a form that is explicit in time**

$$\partial_t f_n + M_{nn}^{-1} (\partial_x v_N + \nu_n) f_n = s_n + M_{nn}^{-1} \sum_{m=0}^{n-1} M_{nm} (s_m - \partial_t f_m).$$

- **In higher dimensionality d, M_{nn} generalizes to a $d \times d$ matrix**
- **For the magnetized case, there is only a 2x2 closure for highest level retained for parallel and perpendicular moments $v_{\|}^n, v_{\perp}^{2m}$**

Polarization in Extended Gyrokinetics

Extension of Gyro-kinetics by Dimits PoP 2010

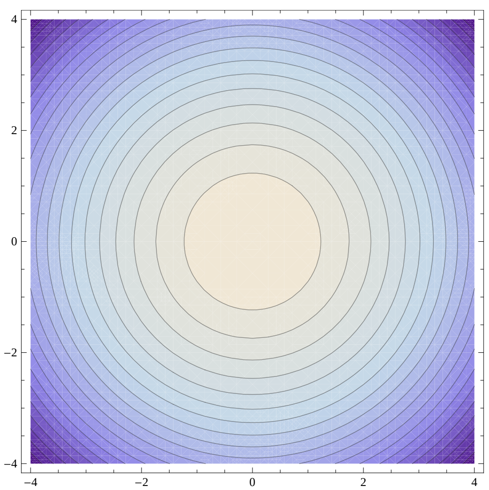
- **Gyrokinetics can be extended to large perturbations**

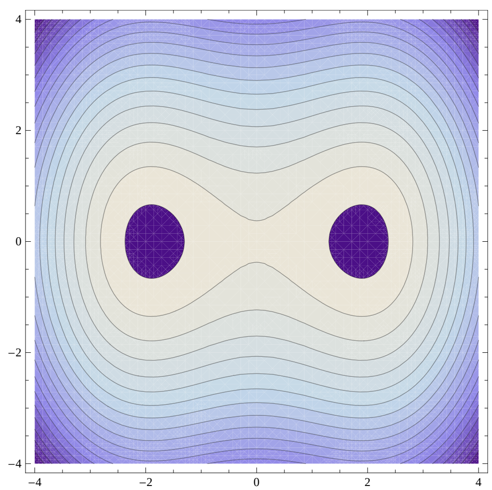
$$e\phi/T \sim 1$$

- **Subject to the restriction that the electric oscillation frequency $\Omega_E^2 = eE'/m$ remain smaller than the gyrofrequency $\Omega = eB/m$**

$$\frac{eE'}{m\Omega^2} = (k\rho)^2 \frac{e\phi}{T} \ll 1$$

  - This coincides with the topological condition that the electric oscillation does not destroy the gyro-orbit
  - Short wavelength still requires $e\Delta\phi/T \ll 1$, but long wavelength can achieve large amplitude





- **This condition defines the applicability limit of ALL gyrokinetic adiabatic expansions**

Non-canonical Action in a Moving Frame

- **Given non-canonical phase space coordinates $\{x, v\}$, the charged particle action form**

$$Ldt = (V(x, v, t) + A(x, t)) \cdot dx - (V(x, v, t)^2/2 + \Phi(x, t))dt$$

- **yields the Euler-Lagrange equations of motion**

$$\begin{aligned} E_v L &= \partial_v V \cdot (\dot{x} - V) = 0 \\ E_x L &= -(\partial_t V + V \cdot \nabla V) + E + V \times B. \end{aligned}$$

- **If we expand $V = v + U(x, v, t)$ then**

$$\begin{aligned} \dot{x} &= v + U \\ A_* &= A + U \\ \Phi_* &= \Phi + U^2/2 \\ (1 + \partial_v U) \cdot \dot{v} &= E_* + V \times B_* \\ &= -V \cdot \nabla U + E + V \times B. \end{aligned}$$

- **Thus, these are the equations of motion of a charged particle moving in reference frame $U(x, v, t)$**

Gyro-Averaged Action in a Moving Frame

- **A magnetized particle travels in gyro-orbits**

$$x = R + \rho(R, u, \mu, t)e^{i\gamma}$$
$$\dot{x} = U(R, u, \mu, t) + \Omega(R, \mu, t)\rho e^{i\gamma} \times \hat{b}$$

- **The Lagrangian becomes**

$$Ldt = (U + \bar{A}) \cdot dR + \mu d\gamma - (U^2/2 + \mu\Omega + \bar{\Phi})dt$$
$$+ (A - \bar{A})dR + (U + \bar{A})d\rho - (\Phi - \bar{\Phi} + U \cdot \Omega\rho e^{i\gamma} \times \hat{b})dt$$
$$+ (A - \bar{A})d\rho$$

- **The lowest order coordinate transformation yields the following definitions**

$$H_0 = U^2/2 + \mu\Omega + \bar{\Phi}$$
$$P_0 = U + \bar{A}$$
$$B_* = \nabla \times (A + U)$$
$$E_* = -\nabla H + \partial_t P_0$$
$$\mathcal{J}_* = \partial_u P_0 \cdot B_* = \partial_u P_0 \cdot \nabla \times (A + U)$$

Gyro-Averaged Equations of Motion

- **The lowest order equations of motion are**

$$\dot{\mu} = 0 \tag{9}$$
$$\dot{R} = U \tag{10}$$
$$U \cdot \partial_u P_0 = \partial_u H_0 \tag{11}$$
$$\dot{u}\partial_u P_0 = E_* + U \times B_* = -\partial_t U - U \cdot \nabla U + E + U \times B \tag{12}$$
$$\dot{\gamma} = \partial_\mu H_0 - U \cdot \partial_\mu P_0. \tag{13}$$

- **Their solution defines the lowest order Poisson bracket**

$$\dot{R} = U = \left(E_* \times \partial_u P_0 + \partial_u H_0 B_*\right) / \mathcal{J}_* \tag{14}$$
$$\dot{u} = B_* \cdot E_* / \mathcal{J}_* \tag{15}$$
$$\dot{\gamma} = \partial_\mu H_0 - \left(\partial_u H_0 (B_* \cdot \partial_\mu P_0) + E_* \cdot \partial_u P_0 \times \partial_\mu P_0\right) / \mathcal{J}_* \tag{16}$$

1st order Adiabatic Expansion

- **The adiabatic expansion searches for the canonical transformation**

$$\mathcal{T}_G x = R + \rho e^{i\gamma} + \ldots$$
$$\mathcal{T}_G \dot{x} = V + \Omega\rho e^{i\gamma} \times \hat{b} + \ldots$$

$$\partial_t G + [G, H_0] = \psi_1 == \equiv H_1 - P_1 \cdot U_0 - P_0 \cdot dX_1$$
$$\psi_1 = \Phi - \bar{\Phi} + U_0 \cdot \Omega\rho e^{i\gamma} \times \hat{b}$$

- **The following 2 choices (Dimits & new!) approximately removes the 1st order $e^{i\gamma}$ perturbations**

$$U_{0,Dimits} = \frac{\hat{b}}{B} \times \nabla_\perp J_0 \Phi$$
$$U_{0,New} = \frac{\hat{b}}{B} \times \frac{\nabla_\perp}{\rho|\nabla_\perp|} 2J_1\Phi$$

- **Leaving only**

$$\tilde{\psi}_1 \sim J_2 \phi e^{i2\gamma} + ... = (\rho\rho - 1/2) : \nabla_\perp \nabla_\perp \Phi/2 + ... \quad (17)$$

- **The first order generating function defines the 2nd order Hamiltonian**

$$G_1 = \int e\tilde{\psi} d\gamma/\Omega$$
$$2\psi_2 = [G_1, \psi_1] = \partial_\gamma G \partial_\mu \psi_1 + \partial_u P_0 \cdot \nabla\psi_1 \times \nabla\tilde{G}_1/\Omega$$
$$= 2\partial_\mu \psi_1^2 + \partial_u P_0 \cdot \nabla\psi_1 \times \nabla\tilde{G}_1/\Omega$$

Linear Polarization in Extended Gyro-kinetics

- **Charge density must be defined via pullback**

$$en_i = \int f d^3v = \int T_g^{-1} F dR dv_{\|} d\mu \mathcal{J}_* d\gamma$$

- **B Jacobian $\mathcal{J}_*$ has non-trivial dependence on Φ**

$$\mathcal{J}_* = \partial_u P_0 \cdot \nabla \times (A + U) = 1 + \nabla \cdot \hat{b} \times U_0 / B^2$$

- **New terms: Dimits form**

$$\begin{aligned}
\mathcal{J}_* &= 1 + \nabla_\perp \cdot (\Omega B)^{-1} \nabla_\perp J_0 \Phi \\
\delta F &= [G_1, F] = -\psi_1 \partial_\mu F - \hat{b} \cdot \nabla G_1 \times \nabla F \\
\overline{\delta F} &= \overline{[G_1, F]} = -(F_0 e / T_\perp) \left[\sum_{n \neq 0} (-1)^n J_n^2 \Phi + J_1 \rho |\nabla_\perp| J_0 \Phi \right] \\
&= -(F_0 / T_\perp) \left(1 - J_0^2 + J_1 J_0 |\nabla_\perp| \rho \right) e\Phi
\end{aligned}$$

- **Linear polarization**

$$n_{i,Dimits} = \Gamma_0^{1/2} n_g + T_{i\perp}^{-1} n_0 \left(\Gamma_0 - 1 - \varrho^2 \nabla_\perp^2 \Gamma_1 \right) \Phi$$

Linear Polarization: New Form

- **New terms: New form**

$$\mathcal{J}_* = 1 + \nabla_\perp \cdot \frac{\nabla_\perp}{B|v_\perp \nabla_\perp|} 2J_1\Phi$$

$$\delta F = [G_1, F] = \psi_1 \partial_\mu F + \hat{b} \cdot \nabla G_1 \times \nabla F$$

$$\overline{\delta F} = \overline{[G_1, F]} = (F_0/T_\perp) \left[\sum_{n \neq 0,1} (J_n^2 e\Phi) \right]$$

$$= -(F_0/T_\perp) \left(1 - J_0^2 - 2J_1^2) e\Phi\right)$$

- **Linear polarization**

$$n_{i,new} = \Gamma_0^{1/2} n_g - 2n_0 T_\perp^{-1} \Gamma_1 e\Phi$$